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Particle sliding on a turntable in the presence of friction

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The motion of a point particle sliding on a turntable is studied. The equations of motion are derived assuming that the table exerts a frictional force on the particle that is of constant magnitude and directed opposite to the particle's motion relative to the turntable. After expressing the equations in terms of dimensionless variables, some of the general properties of the solutions are discussed. Approximate analytic solutions are found for the cases in which (i) the particle is released from rest with respect to the lab frame, and (ii) the particle is released from rest with respect to the turntable. The equations are then solved numerically to get a more complete understanding of the motion. It is found that one can define an escape speed for the particle, which is the minimum speed required to get the particle to move out to infinity. The escape speed is a function of both the distance from the center of the turntable and the direction of the initial velocity. A qualitative explanation of this behavior is given in terms of fictitious forces. Numerical study also indicates an alternative way to measure the coefficient of friction between the particle and the turntable.
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I. INTRODUCTION

The study of particle motion in rotating frames is a crucial part of an undergraduate mechanics course. Earth is a rotating frame and hence the study of any large-scale motion on the surface of the Earth as observed by an Earth-bound observer necessitates the use of a rotating frame. The use of a rotating frame can also simplify the study of certain mechanics problems that involve rotating bodies in the laboratory. One of the first examples discussed in an undergraduate lecture on rotating frames is the motion of a particle on a rotating two-dimensional platform. The turntable is used extensively in setting up demonstration experiments to help students understand the fictitious Coriolis and centrifugal forces.[1–3] A rotating platform on which a particle is moving also forms the basis of a large number of textbook problems elucidating Coriolis and centrifugal forces and the application of the polar coordinate system.

There are a few particle-on-a-turntable problems that are exactly solvable. The easiest is the motion of a particle that is not coupled to the table (sliding without friction). In that case, the motion is one with uniform velocity when observed from the lab frame. This linear motion, when observed from the turntable, is not so simple and one has to invoke Coriolis and centrifugal forces to explain the resulting trajectory.[4,5] A more difficult problem that can be exactly solved is that of a spherical ball moving without slipping on a turntable.[6] The motion in this case is similar to that of a charged particle in a magnetic field. The ball follows a circular trajectory whose location and radius depends on the initial conditions. The variations of this problem where the turntable is tilted[6,7] and freely spinning[8] have also been studied. Solution of the former yields cycloidal motion, and for the latter the trajectories are conic sections. Introducing sliding effects renders the problem difficult to treat analytically. The motion of a ball that rolls with sliding on a turntable has been studied using computer simulations and experiments.[9]

Though most problems involving turntables in physics textbooks deal with friction, they tend to significantly restrict the motion. For example, a typical problem involves finding the maximum angular frequency with which the table can rotate if the particle is to remain at rest with respect to the table at a given distance from the rotation axis. But as we shall see, the same system offers an interesting set of behaviors if one is willing to probe a little further. A detailed study of the general motion under such circumstances is missing in the literature. The present work is an attempt to fill that gap by studying the motion of a point particle on a turntable in the presence of friction, both analytically and numerically.

In Sec. II, the equations of motions for the sliding particle are derived and expressed in dimensionless form. Some of the general properties of the solutions to the equations are then discussed. In Sec. III, solutions to the equations are derived for two special cases: (i) when the particles are moving slowly with respect to the lab frame, and (ii) when the particle is moving slowly with respect to the turntable. Section IV deals with the numerical solution of the problem; various trajectories are illustrated and the dependence of the particle escape speed on the initial direction of motion is determined at various locations on the table. In the final section, we conclude with a brief summary and possible extensions of the current work.

II. THE EQUATIONS OF MOTION

Consider a point particle of mass m on a turntable of infinite extent. Let the coefficient of friction between the particle and the table surface be μ (we will assume that the coefficients of static and kinetic friction are the same). Assume that the table is rotating with a uniform angular speed Ω about an axis that is perpendicular to the plane of the table. Let (r, θ) be the polar coordinates of the particle with respect to an inertial frame, defined by taking the point of intersection of the rotation axis and the table as the origin. For a particle at rest with respect to the table, the frictional force on the particle will be in the (negative) radial direction. The magnitude of this static friction force will vary from 0 to μmg, depending on the distance of the particle from the origin. When there is relative motion between the particle and the table, the magnitude of the kinetic friction force on the particle will be μmg. The direction of the frictional force in this case is determined by the velocity of the particle relative to the table:

$$\vec{v}_{\rm rel} = (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) - r\Omega\hat{\theta}. \tag{1}$$

Since the frictional force opposes the relative motion between the particle and the table, its direction will be opposite to that of $\vec{v}_{\rm rel}$. The frictional force on the particle, when it is in motion with respect to the table, is thus given by

$$\vec{F} = \mu mg(-\hat{v}_{\rm rel}) = \mu mg\frac{r\Omega\hat{\theta} - (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{\sqrt{r^2(\Omega-\dot{\theta})^2 + \dot{r}^2}}, \tag{2}$$

where $\hat{v}_{\rm rel}$ is the unit vector in the direction of the relative velocity. The equations of motion in polar coordinates then take the form

$$\ddot{r} - r\dot{\theta}^2 = \frac{-\mu g\dot{r}}{\sqrt{r^2(\Omega-\dot{\theta})^2 + \dot{r}^2}} \tag{3}$$

and

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{\mu g r(\Omega-\dot{\theta})}{\sqrt{r^2(\Omega-\dot{\theta})^2 + \dot{r}^2}}. \tag{4}$$

As mentioned above, if the particle remains at rest with respect to the table (that is, if $\dot{r} = \ddot{r} = 0$ and $\dot{\theta} = \Omega$), the static frictional force on the particle will be $\vec{F}_s = -m\Omega^2 R\,\hat{r}$, where $r = R$ is the radial location of the particle.

The time scale in the problem is set by the angular velocity of the turntable Ω. There is also an inherent length scale in the problem, $R_{\max}$, the radial distance beyond which the particle cannot remain at rest with respect to the turntable. This distance is found by equating the centrifugal force $m\Omega^2 R_{\max}$ in the rotating frame (at $r = R_{\max}$) to the maximum value μmg of the static friction force, giving

$$R_{\max} = \frac{\mu g}{\Omega^2}. \tag{5}$$

It is convenient to define the two dimensionless variables $\tau \equiv t\Omega$ and $\xi \equiv r/R_{\max}$. Equations (3) and (4) in terms of these new variables are

$$\xi'' - \xi\theta'^2 = \frac{-\xi'}{\sqrt{\xi^2(1-\theta')^2 + \xi'^2}} \tag{6}$$

and

$$2\xi'\theta' + \xi\theta'' = \frac{\xi(1-\theta')}{\sqrt{\xi^2(1-\theta')^2 + \xi'^2}}, \tag{7}$$

where the prime denotes differentiation with respect to τ.

The equations obtained above are two coupled nonlinear equations, and a complete analytic solution looks difficult. The non-conservative nature of the frictional force makes it impossible to write down the first integrals of motion using conservation laws. Nevertheless, one can discern certain general properties of the solution:

1. If the particle is at rest with respect to the table within the region given by $\xi < 1$, it remains at rest with respect to the table. In the rotating frame of the turntable, the particle velocity is zero initially and hence the Coriolis force will be absent at that initial instant. The centrifugal force acting in the radial direction will be balanced by the static friction force because $r < R_{\max}$. In particular, if the particle comes to rest with respect to the table in the internal region given by $\xi < 1$, the particle will subsequently remain at rest with respect to the table.
2. Since $\xi \geq 0$, then from Eq. (6) we find that if $\xi' \leq 0$, then $\xi'' \geq 0$. Thus, if $\xi' < 0$ initially (that is, if the radial velocity points inward), then ξ will reach a minimum value and will thereafter increase monotonically with time. Moreover, if $\xi' > 0$ (the radial velocity points outward), then ξ will increase monotonically with τ; this is because if ξ' ever becomes zero, it can only become positive (or remain zero) in the next instant (since $\xi'' \geq 0$). However, in both cases, if $\theta' = 1$ (i.e., $\dot{\theta} = \Omega$) when $\xi' = 0$ and $\xi < 1$, then the particle will come to rest in the frame of the table.
3. From Eq. (7), we see that if $\theta' = 0$, then $\theta'' > 0$. This implies that if one starts with $\theta' > 0$ (a tangential velocity in the direction of the rotation of the turntable), it will remain so at all future times.
4. From point 2 above, we know that ξ' will eventually become positive (or equal to 0). Then after long enough times—long enough to ensure that ξ' is positive—Eq. (7) shows that if $\theta' \geq 1$, then $\theta'' < 0$. Under similar conditions (with $\xi' > 0$), we see that if $\theta' < 0$, then $\theta'' > 0$. These two conditions ensure that eventually the value of θ' lies between 0 and 1.

The conclusions are what one would expect intuitively. If the particle does not come to rest with respect to the table, it will eventually move away from the center to greater distances. And its angular velocity, as observed from the lab frame, will eventually lie between 0 and angular velocity of the table Ω.

III. ANALYSIS FOR SPECIAL CASES

Two cases where one can approximate the equations of motion so that analytical solutions are possible are discussed below. These are: (A) when the particle moves slowly as observed in the lab frame, and (B) when the particle moves slowly as observed in the rotating frame.

A. Particle moving slowly in the lab frame

Imagine that the particle has been placed gently on to the turntable by a person who is at rest in the lab frame. This would mean that the particle velocity is zero initially and it will remain small for a certain period of time at least. The particle velocity can be considered to be small if $|\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}| \ll \Omega r$; that is, when $\dot{r} \ll \Omega r$ and $\dot{\theta} \ll \Omega$. In terms of the dimensionless variables, these conditions become $\xi' \ll \xi$ and $\theta' \ll 1$. If the initial condition is such that $\xi'(0) \ll \xi(0)$ and $\theta'(0) \ll 1$, there will be a time range in which ξ' and θ' remain small compared to ξ and 1, respectively. During this time range, we can approximate $(1-\theta') \approx 1$ in the equations of motion, so that Eqs. (6) and (7) become

$$\xi'' = \xi\theta'^2 - \frac{\xi'}{\sqrt{\xi^2 + \xi'^2}} \tag{8}$$

and

$$\theta'' = \frac{1}{\sqrt{\xi^2 + \xi'^2}} - \frac{2\xi'\theta'}{\xi}. \tag{9}$$

To simplify further, we neglect ξ' in comparison to ξ and replace ξ with $\xi(0) \equiv \xi_0$, the value of ξ at $\tau = 0$. The resulting equations will be valid as long as ξ', which is initially small, has not grown to be comparable to ξ, and while ξ itself has not changed appreciably from its initial value. These approximations will need to be taken into account when estimating the range of validity of the following analysis (see below).

Under these assumptions, the radial equation is

$$\xi'' = \xi_0\theta'^2 - \frac{\xi'}{\xi_0} \tag{10}$$

and the tangential equation becomes

$$\theta'' = \frac{1}{\xi_0}. \tag{11}$$

Integration of Eq. (11) with initial conditions $\theta(0) = 0$ and $\theta'(0) = 0$ gives

$$\theta = \frac{p^2}{2\xi_0}. \tag{12}$$

(It is always possible to choose $\theta(0) = 0$ simply by defining θ appropriately.) Substituting the solution for θ into the radial equation gives a second-order inhomogeneous equation for ξ. The solution with initial conditions $\xi(0) = \xi_0$ and $\xi'(0) = 0$ gives

$$\xi = \frac{p^4}{12\xi_0} + \xi_0. \tag{13}$$

The approximations made in arriving at the above solution are $\theta' \ll 1$, $\xi' \ll \xi$, and $|\xi - \xi_0| \ll \xi_0$. For the solution obtained these conditions reduce to $\tau \ll \xi_0$, $\tau \ll \xi_0^{2/3}$, and $\tau \ll \sqrt{\xi_0}$, respectively. The last of these conditions is the one that gets violated first as time progresses.

Figure 1 shows the trajectories obtained by numerical integration of the exact equations of motion, compared to the approximate analytic solutions. The numerical work was carried out in MATLAB and is discussed in greater detail in Sec. IV. For the trajectories shown in Fig. 1 we used $\xi_0 = 500$ and placed the particle at rest in the lab frame at $\tau = 0$. The agreement between the numerical and analytical solutions is found to be good. The solution should be valid until $\tau \lesssim \sqrt{500} \approx 22$ and this is approximately where the curves start deviating from the numerical results.

B. Particle moving slowly with respect to the turntable

Consider the case where the particle is placed at some location on the turntable with zero initial velocity with respect to the rotating frame. The particle velocity with respect to the turntable will then be small at least for some interval of time. Making the relevant approximations, we can analytically solve for the motion of the particle during this time.

When the particle is moving slowly with respect to the turntable, we have $\theta' \approx 1$ and $\xi' \approx 0$. Define

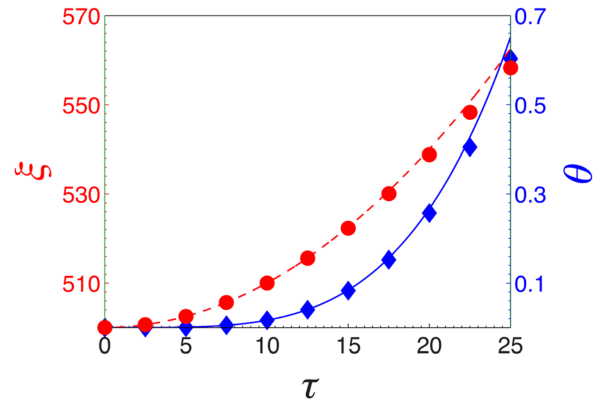

Fig. 1. Comparison of our approximate analytic solution with numerical calculations for the motion of a particle released from rest with respect to the lab frame at $\xi_0 = 500$. The analytic predictions for $\xi(\tau)$ (dashed curve, left axis) and $\theta(\tau)$ (solid curve, right axis) closely match the the numerical results (circles for ξ and diamonds for θ). The time τ is measured in units of $1/\Omega$, while ξ is measured in units of R_{max}.

$$\Theta' \equiv \theta' - 1, \tag{14}$$

which is the angular velocity of the particle in the rotating frame. The condition that $\theta' \approx 1$ implies that $|\Theta'| \ll 1$. Neglecting the term $\xi\Theta'$ compared to ξ in the radial equation and $2\xi'\Theta'$ compared to $2\xi'$ in the tangential equation, the equations of motion become

$$\xi'' - \xi = \frac{-\xi'}{\sqrt{\xi^2\Theta'^2 + \xi'^2}} \tag{15}$$

and

$$2\xi' + \xi\Theta'' = -\frac{\xi\Theta'}{\sqrt{\xi^2\Theta'^2 + \xi'^2}}. \tag{16}$$

Since $\xi' = \Theta' = 0$ initially, the term $\xi^2\Theta'^2$ can be discarded in comparison to ξ'^2 inside the square root in both equations. The justification for this approximation is that ξ' grows faster with time than $\xi\Theta'$. As the particle starts from rest in the frame of the turntable, the velocity-independent centrifugal force tends to increase the radial velocity whereas the tangential velocity buildup can happen only once the velocity-dependent Coriolis force is appreciable. Under this additional approximation, Eqs. (15) and (16) become

$$\xi'' - \xi = -1 \tag{17}$$

and

$$2\xi' + \xi\Theta'' = 0, \tag{18}$$

where in the second equation the term $\xi\Theta'/\xi'$ has been neglected. The resulting equation for the radial motion [Eq. (17)] is identical to the equation for radial motion of a bead on a uniformly rotating rod in the presence of friction. The solution to this equation is

$$\xi = (\xi_0 - 1)\cosh\tau + 1, \tag{19}$$

where the boundary conditions $\xi(0) = \xi_0$ and $\xi'(0) = 0$ have been used to determine the constants of integration. Note that the above solution is valid only when $\xi_0 > 1$. If $\xi_0 < 1$, the particle will remain at rest with respect to the table. In the limit of ξ_0 approaching 1, the equation predicts $\xi = 1$ for all times, as expected.

Integrating Eq. (18) once and using the boundary conditions above, we obtain

$$\Theta' = 2 \log\left(\frac{\xi_0}{\xi}\right). \tag{20}$$

The solution obtained for ξ can be substituted into this equation and integrated to get the variation of Θ with time. This solution is valid provided $|\Theta'| \ll 1$ and $|\xi\Theta'| \ll |\xi'|$. For values of $\xi_0 \gg 1$, these conditions are satisfied for $\tau \ll 1$. For values of ξ_0 closer to (and greater than) 1, the approximations hold for longer times. A comparison of the solutions from this approximate analytical analysis and the exact numerical results is shown in Fig. 2.

IV. NUMERICAL ANALYSIS

The analytic analysis carried out above is limited in its scope and it does not yield the rich variety of trajectories that are possible in this system. In this section, we present numerical solutions to the exact equations and classify the resulting trajectories in terms of their long-time behavior.

The equations of motion [see Eq. (2)] expressed in Cartesian coordinates are

$$\begin{aligned} m\dot{v}_x &= \mu m g \frac{-\Omega y - v_x}{\sqrt{(\Omega y + v_x)^2 + (\Omega x - v_y)^2}}, \\ m\dot{v}_y &= \mu m g \frac{\Omega x - v_y}{\sqrt{(\Omega y + v_x)^2 + (\Omega x - v_y)^2}}, \\ \dot{x} &= v_x, \\ \dot{y} &= v_y. \end{aligned} \tag{21}$$

These four coupled differential equations have been solved using the second-order Runge-Kutta method for a variety of initial conditions. The unit of time is chosen such that the magnitude of Ω is 1 and distances are measured in units of $R_{\max}$. This allows comparison to be made between the numerical results and the analytical results above (Figs. 1 and 2). The time step in the simulation was taken to be $\delta t = 0.001$; this was found to be sufficient for convergence of the trajectories to acceptable limits for the entire time duration for which trajectories were generated.

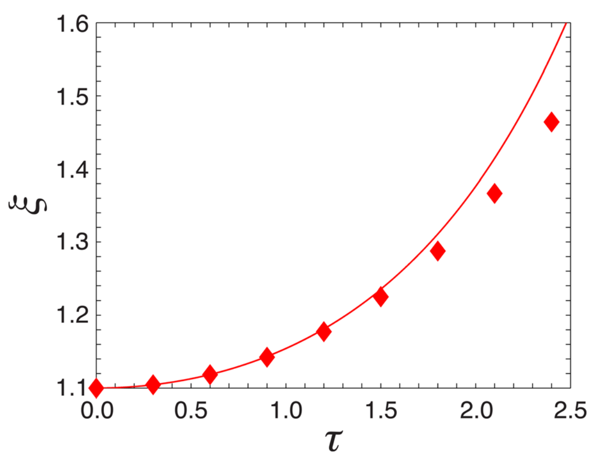

Fig. 2. Comparison of the theoretical prediction for $\xi(\tau)$ with numerical results for a particle released at $\xi_0 = 1.1$ with zero velocity with respect to the turntable. The analytic solution (solid curve) matches the numerical results (diamonds) for times $\tau \lesssim 1$.

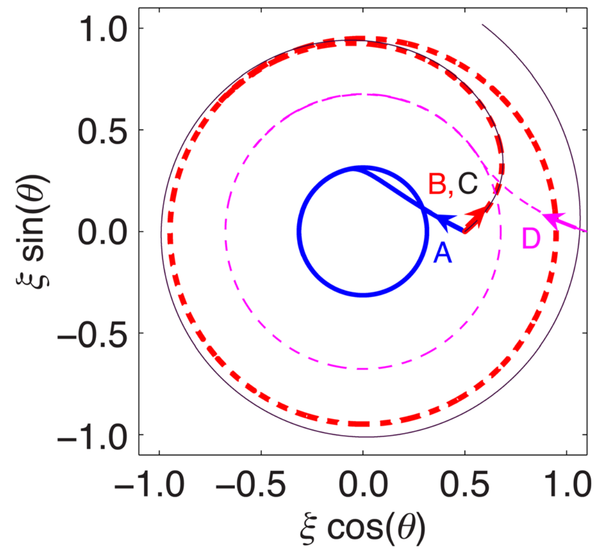

Fig. 3. Trajectories of the particle as seen from the lab frame for four different initial conditions. The parameters varied are ξ_0 (initial location), ϕ (the direction of the initial velocity measured in the turntable's frame with respect to the radially outward direction), and v_0 (the initial speed). The arrows indicate the starting location and initial direction. The table is rotating counterclockwise as seen from the top. The initial conditions are: $\xi_0 = 0.5$, $\phi = \pi$, and $v_0 = 1$ for trajectory A (thick, solid curve, blue online); $\xi_0 = 0.5$, $\phi = 0$, and $v_0 = 0.6000$ for trajectory B (thick, dashed curve, red online); $\xi_0 = 0.5$, $\phi = 0$, and $v_0 = 0.6078$ for trajectory C (thin, solid black curve); and $\xi_0 = 1.1$, $\phi = 1.2\pi$, and $v_0 = 1.3$ for trajectory D (thin, dashed curve, pink online).

Figure 3 shows some of the particle trajectories obtained from the simulations, as seen from the lab frame; Fig. 4 shows the same trajectories as seen from the rotating frame. One can see that the trajectories exhibit a variety of behavior. The trajectory depends on the point at which the particle is released, the speed with which it is released, and the direction of the initial velocity. In trajectories A and B (see figures), the particle starts off in the region $\xi < 1$ and comes to rest with respect to the table. In trajectory C, which has almost the same initial conditions as trajectory B (except for a marginal increase in the initial speed), the particle manages to come out of the $\xi = 1$ region and eventually moves off to

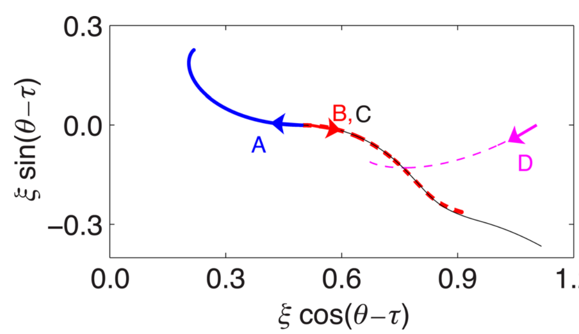

Fig. 4. The same trajectories as in Fig. 3, as seen by an observer in the rotating frame. The arrows indicate the starting location and direction of the initial velocity. In all cases except C (thin, black, solid curve), the particle eventually comes to rest with respect to the turntable.

infinity. Trajectory D shows another possibility, where the particle is launched from the region $\xi > 1$, but the initial conditions are such that it comes to rest with respect to the table in the region $\xi < 1$. One can classify these trajectories into those executing bounded motion and those that move off to infinity. As discussed in Sec. II, the bound trajectories all correspond to the particle coming to rest with respect to the turntable in the region $\xi < 1$.

It is interesting that this system exhibits an *escape* speed—a minimum initial speed required to get the particle to move off to infinity. From the simulations, the escape speed v_e is found to depend on the location where the particle is released as well as the direction of its initial velocity. This is unlike the case for motion in a conservative central force field where the escape speed is independent of the direction of the velocity. Figure 5 shows the variation of the escape speed as a function of the initial direction ϕ (as measured by an observer on the turntable with respect to the radially outward direction), for several values of the initial position, ξ_0. The minimum escape speed occurs at $\phi \approx 0.4$ radians for all values of ξ_0 less than 1. Qualitatively this makes sense, because for a noninertial observer on the turntable the Coriolis force tries to deflect the particle to the right of its direction of motion (for the case when the table is rotating counterclockwise as seen from the top), so it should be advantageous to release the particle to the left of the radially outward direction (rather than in the radial direction itself) so as to make it move farther away from the center. For the same reason, it is most difficult to get the particle to move away to infinity if it is released at an angle of about $\phi \approx 4$ radians with respect to the outward radial direction. In this case, the Coriolis force tries to push the particle towards the center of the table, confining it more to the interior region.

The variation of escape speed with ϕ is more complicated when the particle starts outside the $\xi < 1$ region and a complete analysis will not be presented here. Although a particle left at rest with respect to the table at $\xi > 1$ will move off to infinity, it is possible to get the particle to come rest in the region $\xi < 1$ for a range of initial speeds that depend on the angle ϕ (trajectory D in Fig. 3, for example).

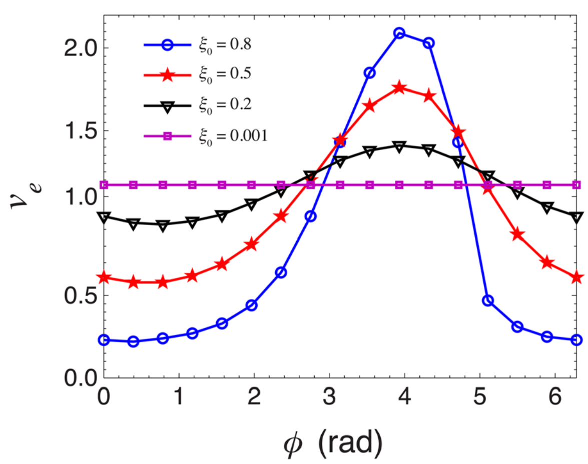

Fig. 5. Escape speed v_e as a function of launch angle ϕ. Both the speed and the angle are as seen by an observer located at the point of release of the particle on the turntable and at rest with respect to the turntable. The angle $\phi = 0$ corresponds to the outward radial direction. The four data sets are for release at $\xi_0 = 0.8$ (circles), $\xi_0 = 0.5$ (stars), $\xi_0 = 0.2$ (inverted triangles), and $\xi_0 = 0.001$ (squares).

Lastly, let us consider the final position ξ_f of a particle that is placed gently on the table by an inertial observer. In this case, the particle is initially at rest in the lab frame. Figure 6 shows ξ_f as a function of initial radius ξ_0, obtained numerically. The maximum distance at which the particle can be placed so that it will still remain bounded is $\xi_{0c} = 0.87$. Experimental determination of this distance could be a way of finding the coefficient of friction between the particle and the table. Using Eq. (5), we can relate ξ_{0c} to the critical distance r_{0c} in conventional units:

$$\mu = \frac{\Omega^2 r_{0c}}{\xi_{0c}\, g}. \tag{22}$$

This method is to be contrasted with the more familiar one, wherein one would place the particle at a known location r_a with the table initially at rest and find the minimum rotational speed of the turntable at which the particle begins to slide. If $\Omega_{\min}$ is the minimum value of the angular speed for a distance r_a, then

$$\mu = \frac{\Omega_{\min}^2 r_a}{g}. \tag{23}$$

Although these two expressions for μ look similar, the underlying concepts employed are subtly different.

V. SUMMARY

A particle sliding on a rotating table exhibits interesting behaviors that are overlooked in the typical textbook treatment. The motions of the particle for short durations after it has been released from rest (both with respect to the turntable and with respect to the lab) can be found analytically. Numerical integration of the exact equations yields a rich array of behaviors over longer time periods.

All the results we have discussed in the paper should be verifiable in an undergraduate lab. It must be kept in mind that our analysis assumes that the particle is point-like. The effects of nonzero particle size could be important, as is the case, for example, in the motion of a puck subject to friction on a stationary surface.[10,11] There are two types of such

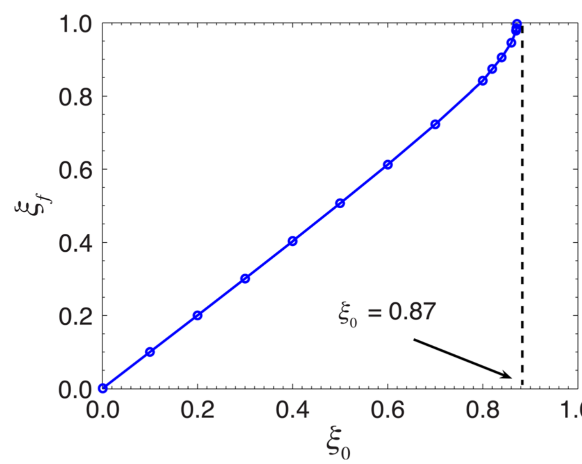

Fig. 6. The final location of the particle as a function of the initial distance from the center, for a particle released at rest with respect to the lab frame. When $\xi_0 > 0.87$, the particle moves off to infinity.

effects: the tendency of the object to topple, and its tendency to rotate about an axis perpendicular to the plane of the turntable. The first can be mitigated by using an object, whose height is much less than its lateral dimension. The second effect will not alter the results appreciably so long as the particle's distance from the center of the turntable is large compared to its lateral size (see Appendix). Thus, a sufficiently small object on a rotating table should show a behavior similar to the predictions made in this paper.

A natural extension to the work presented here would be to study the motion that results when the coupling between the particle and the table is not of the dry-friction type. Preliminary analysis of Stokes' drag coupling indicates a completely different kind of dynamics.

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APPENDIX

The calculations above assume that the particle is point-like. Here, we use an order-of-magnitude analysis to determine the conditions required for this assumption to be valid.

First, consider the tendency for the particle to topple. Suppose that the particle has a lateral size L and height h, the exact shape being immaterial for the analysis that follows. Assuming that the particle is undergoing pure translational motion, the forces acting on it are gravity, the normal force, and the friction force pointing opposite the direction of the particle's speed with respect to the table. To find when the particle would topple, we need to compute the component of the torque about the center-of-mass that lies in the plane of the table. To make the analysis simple, we assume that the reaction forces act at two points on the surface of contact that are located at distances $L/4$ on either side of the center-of-mass along the direction of the relative velocity of the body with respect to the turntable (see Fig. 7). The condition for rotational equilibrium about an axis passing through the center-of-mass and parallel to the surface is

$$\mu mg\frac{h}{2}+N_1\frac{L}{4}-N_2\frac{L}{4}=0. \tag{A1}$$

The condition that the center-of-mass not move in the vertical direction gives

$$N_1+N_2=mg. \tag{A2}$$

Solving the above equations gives

$$N_2=\frac{mg}{2}+\mu mg\frac{h}{L}. \tag{A3}$$

Requiring $N_1 \approx N_2$ (otherwise parts of the body will tend to lose contact with the table) gives the condition $\mu h/L \ll 1$. Since μ is typically of order 1 or less, the above condition will be satisfied provided $h \ll L$.

Rotation about an axis parallel to Ω (that is, in the plane of the turntable) can happen if there is a net torque in the vertical direction. Even when it is purely translating, a body of nonzero size has different velocities, at its different parts, with respect to the turntable. As a result, the frictional force acting at different points of contact can be pointing in different directions, leading to a nonzero torque about the center-of-mass; this can in turn lead to an in-plane rotation of the body, even if it starts with pure translational motion. If the linear velocities of the parts of the object due to this rotation

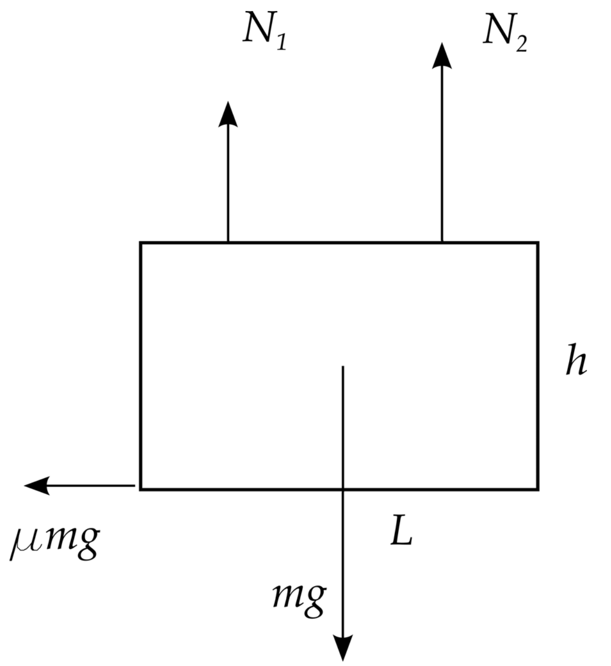

Fig. 7. Free-body diagram for a body of height h, lateral size L, and mass m, purely translating on the turntable. The forces acting on the body are indicated.

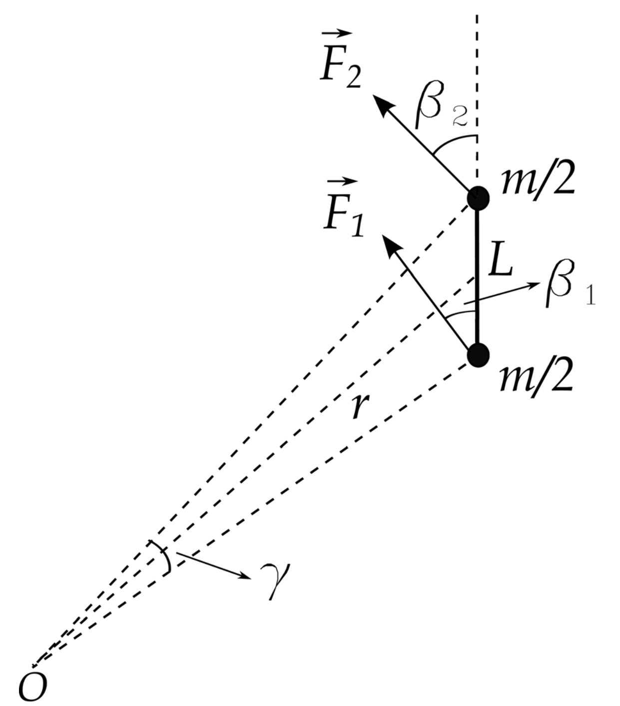

Fig. 8. A rigid body consisting of two point masses, each of mass $m/2$, separated by a massless rigid rod of length L. The body is located at a distance r from the center of the turntable and subtends an angle γ. Vectors $\vec{F}_1$ and $\vec{F}_2$ are the frictional forces acting on the masses and these forces make angles β_1 and β_2 with the connecting rod.

become comparable to the translational velocity of the center-of-mass, the equations of motion used above will not yield accurate results.

To get an order-of-magnitude estimate of when the size of the particle will modify the equation of motion for the center-of-mass [Eq. (2)], consider a body consisting of two point particles of mass $m/2$ each, rigidly attached with a massless rod of length L and located at a distance r from the center of the turntable (see Fig. 8). Assuming the body to be at rest with respect to the lab frame, the torque on the body is

$$\tau = \mu \frac{mg}{2}\frac{L}{2}(\sin\beta_2 - \sin\beta_1), \tag{A4}$$

where the different angles are indicated in Fig. 8. But $\beta_2 = \beta_1 + \gamma$ and γ is of order L/r, provided $L \ll r$. Substituting and simplifying, one gets

$$\tau \lesssim \frac{\mu mgL^2}{r}, \tag{A5}$$

where $\cos\gamma \approx 1$, $\sin\gamma \approx L/r$, and $\cos\beta_1 \lesssim 1$ have been used to get an order-of-magnitude value. The angular acceleration α produced by this torque is obtained by dividing the torque by the moment of inertia about the center-of-mass $I = mL^2/2$, giving

$$\alpha \lesssim \frac{\mu g}{r}. \tag{A6}$$

The linear acceleration of the masses corresponding to this angular acceleration is

$$a \sim \mu g \frac{L}{r}. \tag{A7}$$

On the other hand, the net force acting on the center-of-mass is of order μmg (assuming $\cos\gamma \approx 1$) and the corresponding linear acceleration of the center-of-mass is

$$a' \sim \mu g. \tag{A8}$$

Comparing a and a', it is seen that the effects of rotation of the body can be neglected provided that $L \ll r$, for an object that was initially at rest in the lab frame.

Thus, the two conditions required for the body to be treated as a point particle are that its lateral size be much larger than its height and that its distance from the center of the table be large compared to its lateral size. These conditions are only approximate. For example, one could have an initial condition in which rotational motion is of such a magnitude that the frictional force acting at different locations cannot be treated as being collinear. Nevertheless, this analysis gives us a ballpark figure of when the results derived in the paper are valid.

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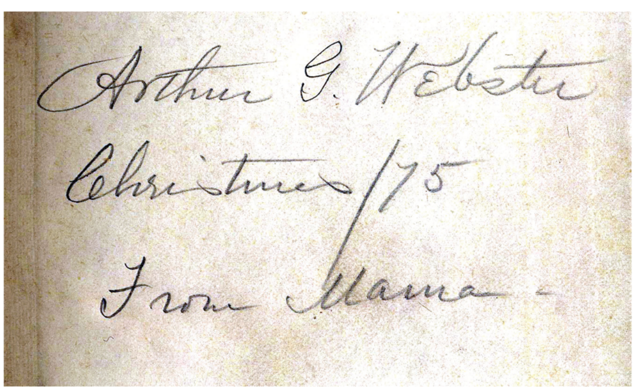

Arthur G. Webster

This is the inscription on the flyleaf of my copy of the "new edition" of John Henry Pepper, *The Boy's Playbook of Science* (George Routledge and Sons, London, ca. 1874). Arthur Gordon Webster (1863-1923) apparently liked this Christmas present from his mother, for he spent his career as the professor of physics at Clark University in Worcester, Massachusetts. He graduated from Harvard and in 1890 received his doctorate from the University of Berlin after working with Herman Helmholtz. At Clark he succeeded Albert Michelson when the latter went to the University of Chicago. Although he wrote several well-known physics and mathematics texts, he is probably best known for being one of the founders of the American Physical Society in 1899. (Notes by Thomas B. Greenslade, Jr., Kenyon College)